# THE MORSE'S LEMMA AND SOME OF ITS APPLICATIONS

***Alimov S.Kh.***
*Jizzakh branch of National University of Uzbekistan,*
*Associate professor at the Department of Applied Mathematics*
***Nurmonova E'.U.***
*Jizzakh branch of National University of Uzbekistan, Student*
***Bakhriddinova A.D.***
*Jizzakh branch of National University of Uzbekistan, Student*
***Almuratov F.M.***
*Jizzakh branch of National University of Uzbekistan,*
*Associate professor at the Department of Applied Mathematics*

**Tayanch so'zlar:** Morse lemmasi, matematik tahlil, silliq funksiyalar, kritik nuqtalar, birlik nazariyasi.

**Ключевые слова:** лемма Морса, математический анализ, гладкие функции, критические точки, теория особенностей.

**Key words:** Morse's Lemma, mathematical analysis, smooth functions, critical points, singularity theory.

Introduction.Today, science is developing rapidly, especially after the integration of computer technologies into science, it is developing several times faster. Scientific innovations and scientific discoveries are being made every day.Including, in the field of mathematics,new theorems and their applications are being studied every day.

We will analyze Morse`s lemma about normalization of a function with an eigencritical point around this point, and it should be noted that this lemma is of great importance for the development of the field. In the work of this article, we study Morse`s lemma and some of its a applications. Usually, in many problems of analysis, the derivative of a smooth function is considered around non-zero points. In this case,undisclosed functions are solved using the theorem in most cases. But if we are at the critical point of the function, we cannot use undisclosed functions. Such problems are, for example, evaluating the order of integration of the function in oscillating integrals of the Fourier image of the function. We use Morse`s lemma to obtain the asymptotic expansions of some integrals. Morse`s lemma is one of the important proofs of the differential theory of making a real-valued smooth function

Into a canonical form around its eigencritical point. The purpose of this dissertation is to study the importance and applications of Morse`s lemma. Therefore, we will study the necessary rates and theorems. Morse`s lemma consists of representing a function in the form of a quadratic form around a fixed critical point. We know that according to Taylor`s formula at the critical point

$$F(x) = f(x) - f(x_0) = \frac{1}{2!}\sum_{i,j}\frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)\left(x^i - x_0^i\right)\left(x^i - x_0^j\right) + 0(||x - x_0||^2)$$

If $o\left(||x - x_0||^2\right)$ is not a residual term on the right side of the equation, that is, $f(x) - f(x_0)$ is a differential quadratic form, then it is known from algebra that it can be reduced to the canonical form indicated by linear transformation. At the same time, we are also based on the inverse function theorem and Adamard`s lemma.

## 2. PROOF OF HADAMARD'S LEMMA

**Definition 1.** $f'(x_0) = 0$ the point $x_0$ where the equality holds is called the critical point of the f(x) function. Where, the deravative means vector. For example:

$$\frac{\partial f}{\partial x^1}(x_0^1), \frac{\partial f}{\partial x^2}(x_0^2), \dots, \frac{\partial f}{\partial x^n}(x_0^n))$$

**Definition 2.** $x_0$ *let the point be the critical point of the function* $f \in C^2(U;R)$. *It is a neighborhood of the point* $-x_0$. *If the determinant of the Hessian of the function at the point*

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^i}\right)$$

*that is, the matrix consisting of second-order special derivatives) is different from zero, than the point* $x_0$ *is called a nondegenerate critical point.*

**Lemma 3. [Hadamard's lemma.]** $f: U \to R$ *the function* $C^{(p)}(U,R)$, $p \geq 1$, *let* $0 = (0,0,0,\dots\dots) \in R^m$ *be the circle of the point and the equality* $f(0) = 0$ *is valid. Than there are such functions* $g_i \in C^{(p-1)}(U,R)$, $(i = 0,1,2,\dots\dots,m)$ *that the relation*

$$f(x^1, x^2, \dots\dots, x^m) = \sum_{i=1}^{m} x^i g_i(x^1, x^2, \dots\dots, x^m) \qquad (1)$$

*is fulfilled for all* $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ *and also the equality holds.*

**Proof.** To prove formula (1), we use the following equality known from mathematic analysis

$$\int_0^1 \phi'(t)dt = \phi(t)\Big|_0^1 = f(x^1, x^2, \dots, x^m) - f(0) = f(x^1, x^2, \dots, x^m), \qquad (2)$$

so

$$f(x^1, x^2, \dots, x^m) = \int_0^1 \frac{df(tx^1,\ tx^2, \dots,\ tx^m)}{dt}dt$$

$$= \int_0^1 \left(\frac{\partial f(tx^1,\ tx^2, \dots,\ tx^m)}{\partial x^1}x^1 + \frac{\partial f(tx^1,\ tx^2, \dots,\ tx^m)}{\partial x^2}x^2 + \cdots\right.$$

$$\left. + \frac{\partial f(tx^1,\ tx^2, \dots,\ tx^m)}{\partial x^m}x^m\right)dt = \int_0^1 \left(\sum_{i=1}^{m}\frac{\partial f(tx^1,\ tx^2, \dots,\ tx^i)}{\partial x^i}x^i\right)dt$$

$$= \sum_{i=1}^{m}\int_0^1 \frac{\partial f(tx^1,\ tx^2, \dots,\ tx^m)}{\partial x^i}x^i dt = \sum_{i=1}^{m} x_i \int_0^1 \frac{\partial f(tx^1,\ tx^2, \dots,\ tx^m)}{\partial x^i}dt,$$

if we define as follows

$$g_i(x^1, x^2, \dots, x^m) = \int_0^1 \frac{\partial f(tx^1,\ tx^2, \dots ,\ tx^m)}{\partial x^i} dt,$$

then we get the equation (1)

$$f(x^1, x^2, \dots, x^m) = \sum_{i=1}^{m} x^i g_i\,(x^1, x^2, \dots, x^m).$$

Now we $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ show the equality

$$g_i(0) = \int_0^1 \frac{\partial f}{\partial x^i}(tx^1, tx^2, \dots, tx^m)dt = \int_0^1 \frac{\partial f}{\partial x^i}(0)dt = \frac{\partial f}{\partial x^i}(0)\int_0^1 dt = \frac{\partial f}{\partial x^i}(0).$$

We show that $g \in C^{(p-1)}(U; R) when\ f \in C^{(p)}(U; R)\ is\ at\ the\ end.$

$$g_i(x^1, x^2, \dots x^m) = \int_0^1 \frac{\partial f(tx^1, tx^2, \dots, tx^m)}{\partial x^i} dt = \left| x_i = \frac{y_i}{t} \right| = \int_0^1 \frac{\partial f(y^1, y^2, \dots \dots, y^m)}{\partial y^i} tdt$$

$$= \{\text{if we integrate by pieces}\}$$

$$= \ tf(y^1, y^2, \dots \dots, y^m) - \int_0^1 f(y^1, y^2, \dots \dots, y^m)\, dt$$

Which shows that $g \in C^{(p-1)}(U{:}\, R)$.

## 3. PROOF OF MORSE'S LEMMA

In this paragraph, we present a Morse lemma and its proof

**Lemma 4. [Morse`s lemma]** *Let the function* $f{:}\, G \to R$ *be defined in class* $C^{(3)}$*(U, R), and let* $G \in R^m$ *and* $x_0 \in G$ *be fixed critical points. Then there exists a* $g{:}\, V \to U$ *diffeomorphism such that* $V - 0 = (0,0,0,\dots) \in R^m$ *is a neighborhood of the point, and the relation* $U - x_0$ *holds for all* $y \in V$

$$(f \circ g)(y) = f(x_0) - [(y^1)^2 + \cdots (y^k)^2] + [(y^{k+1})^2 + \cdots (y^m)^2] \quad (5)$$

*attitude is appropriate.*

**Proof.** $x_0 = 0$ and $f(x_0) = 0$ it is enough for us to prove the case. If $x_0 \neq 0$, If so $x = y + x_0$ if we take a linear substitution like $y = 0$ we can get $(x_0 = y + x_0 \Rightarrow y = 0)$, and this shows that it is enough to prove the case $x_0 = 0$ to prove the point. If $f(x_0) \neq 0$ than it suffices to take, a linear permutation such that $f(x) = \omega(x) + f(x_0)$ where $\varphi(x_0) = 0$ $(f(x_0) = \varphi(x_0) + f(x_0) \Rightarrow \varphi(x_0) = 0)$.

There fore, the point $x_0 = 0$ is the critical point of the function $(f(x))$

$f(x_0) = f(0) = 0$ is equal. Since V – 0 is the circumference of the point and $f \in C^{(3)}(G; R)$, we apply Adamar`s lemma to the function

$$f(x^1, x^2, \dots \dots, x^m) = \sum_{i=1}^{m} x^i\, g_i(x^1, x^2, \dots, x^m)$$

and

$$g_i(0) = \frac{\partial f}{\partial x^i}(0) = 0$$

relations are appropriate. Then we apply Adamard`s lemma to the functions

$$g_i(x^1, x^2, \dots \dots, x^m) = \sum_{j=1}^{m} x^j h_{ij}\,(x^1, x^2, \dots \dots, x^m).$$

and

$$h_{ij}(0) = \frac{\partial g_i}{\partial x^j}(0)$$

In that case the relations

$$g_i(x^1, x^2, \dots., x^m) = \sum_{i,j=1}^{m} x^i x^j h_{ij}(x^1, x^2, \dots., x^m)$$

and

$$h_{ij}(0) = \frac{\partial g_i}{\partial x^j}(0)$$

are reasonable. Thus, we get the equations

$$f(x^1, x^2, \dots, x^m) = \sum_{i,j=1}^{m} x^i\, x^j h_{ij}(x^1, x^2, \dots., x^m) \qquad (6)$$

and

$$h_{ij}(0) = \frac{\partial g_i}{\partial x^j}(0) = \frac{\partial^2 f}{\partial x^i \partial x^j}(0). \qquad (7)$$

We get the equalities.

By replacing $\hat{h}_{ij} = \frac{1}{2}(h_{ij} + h_{ji})$, we bring the the matrix $h_{ij}$ into a symmetric matrix, that is $h_{ij} = h_{ji}$.

$$(x_i x_j h_{ij} + x_j x_i h_{ji} = x_i x_j 2h_{ij} = x_i x_j h_{ij} + x_j x_i h_{ji})$$

Moreover, the point 0 is an invariant critical point for $h_{ij}$. There fore, the determinant of the Hessian matrix $h_{ij}(0) = \frac{\partial^2 f}{\partial x^i \partial x^j}(0)$ is non-zero.

It is possible to convert the $h_{ij}(0)$ matrix into a diagonal matrix by substitutions.

$$\begin{pmatrix} \pm 1 & . & \cdot & \cdots \\ 0 & \dots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} \text{ since it is a symmetric matrix } \begin{pmatrix} \pm 1 & 0 & 0 & \dots \\ 0 & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

then if we continue this process

$$\begin{pmatrix} \pm 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \pm 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & \pm 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & \ddots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

So, in order to make it diagonal as above, we can replace it as follows. $0 \in R^m$ has such $u^1, u^2, \dots., u^m$ coordinates around the point $U_1$ that if we replace $x = \varphi(u)$

$$(f^\circ \varphi)(u) = \pm(u^1)^2 \pm (u^2)^2 \pm \cdots \pm (u^{r-1})^2 + \sum_{i,j=1}^{m} u^i u^j H_{ij}\,(u^1, u^2, \dots u^m) \quad (8)$$

We get the equality. Here $u^1, u^2, \dots, u^m$ coordinates are r ≥ 1 and $H_{ij} = H_{ji.}$If $r = 1$ in (8) then $H_{ij} = H_{ji.}$

The quadratic form in formula (5) is unchanged, that is $det\left(h_{ij}(0)\right) \neq 0$. $x = \varphi(u)$ becomes $\varphi(u)$ since $det\,(\varphi`(0)) \neq 0$ in the substitution is a diffemorphism. In that case, at least one of the numbers $H_{ij}(0)(i,j = r, \dots, m)$ in formula (8) is different from zero. We assume that $H_{rr}(0) \neq 0$, then, from the continuity of the function $H_{ij}(u)$ around the point $u = 0$,the relation $H_{rr}(u) \neq 0$ is valid.

We will prove the rest of the process by the induction method ,that is if we can extract $\pm(u^r)^2$ from the sum in formula (8) ,we will prove the lemma that this process can be continued up to m.

We take $\psi(u^1,\dots,u^m)=\sqrt{|H_{rr}(0)|}$ . Then $\psi\in C^{(1)}(U_2;R)$, is the circumference of the point $U_2\subset U_1-u=0$ here. We change the coordinates as follows.

$$v^i=u^i \qquad i\neq r$$

$$v^r=\psi(u^1,u^2,\dots,u^m)\left(u^r+\sum_{i>r}\frac{u^iH_{ir}(u^1,u^2,\dots,u^m)}{H_{rr}(u^1,u^2,\dots,u^m)}\right)$$

so

$$\sum_{i,j=r}^{m}u^iu^j\,H_{ij}(u)=u^ru^rH_{rr}(u^1,u^2,\dots,u^m)+\sum_{i,j=r+1}^{m}u^iu^jH_{ij}(u^1,u^2,\dots u^m)$$

$$=(u^r)^2H_{rr}(u^1,u^2,\dots,u^m)+\sum_{i,j=r+1}^{m}u^iu^jH_{ij}(u^1,u^2,\dots,u^m)$$

$$=(u^1)^2H_{rr}(u^1,u^2,\dots,u^m)+\sum_{j=r+1}^{m}u^ru^jH_{rj}(u)+\sum_{i=r+1}^{m}u^ru^iH_{ri}(u)$$

$$=(u^r)^2H_{rr}(u^1,u^2,\dots,u^m)+2\sum_{j=r+1}^{m}u^ru^jH_{rj}\,(u^1,u^2,\dots,u^m)$$

$$=\{acording\ to\ the\ substitution\ above\}=$$

$$=(u^r)^2\psi^2(u)$$

$$+2u^r\psi(u)\sum_{j=r+1}^{m}\frac{u^jH_{rj}(u)}{\psi(u)}+(\sum_{j=r+1}^{m}\frac{u^jH_{rj}(u)}{\psi(u)})^2-(\sum_{j=r+1}^{m}\frac{u^jH_{rj}(u)}{\psi(u)})^2$$

$$=(u^r\psi(u)+\sum_{j=r+1}^{m}\frac{u^jH_{rj}(u)}{\psi(u)})^2-(\sum_{j=r+1}^{m}\frac{u^jH_{rj}(u)}{\psi(u)})^2$$

$$=\psi^2(u)(u^r+\sum_{j=r+1}^{m}\frac{u^jH_{rj}(u)}{\psi^2(u)})^2-(\sum_{j=r+1}^{m}\frac{u^jH_{rj}(u)}{\psi(u)})^2$$

$$=|H_{rr}(u)|(u^r+\sum_{j=r+1}^{m}\frac{u^jH_{rj}(u)}{|H_{rr}(u)|})^2-(\sum_{j=r+1}^{m}\frac{u^jH_{rj}(u)}{|H_{rr}(u)|})^2$$

$$=\{we\ exchange\ coordinates\}=\pm(v^r)^2-\frac{1}{H_{rr(v)}}(\sum_{i>r}^{m}v^iH_{ri}(v))^2$$

$$=\pm(v^r)^2-\frac{1}{H_{rr}(v)}(\sum_{i>r}^{m}v^iH_{ri}(v))(\sum_{j>r}^{m}v^jH_{rj}(v))=\pm(v^r)^2-\sum_{i,j>r}^{m}v^iv^j\widehat{H}_{ij}(v),$$

here $\widehat{H}_{ij}=\frac{H_{ij}H_{ji}}{H_{rr}}$ is symmetric. Also, since $v=\psi(u)$ , $u=\psi^{-1}(v)$ is

$$(f*\varphi*\psi^{-1})(v)=\sum_{i=1}^{r}[\pm(v^i)]^2+\sum_{i,j>r}^{m}v^iv^j\widetilde{H}_{ij}(v^1,v^2,\dots,v^m),$$

the relation is appropriate. Here $\varphi*\psi^{-1}$ is a diffeomorphism and $\widetilde{H}_{ij}=\widetilde{H}_{ji}$.Therefore. Morse`s lemma has been proved according to the induction method.

## 4. APLICATION OF MORSE’S LEMMA

**Example 1.** Let`s look at this equation:

$$F(x,y)=0$$

For example, this one:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Let\`s take Eq.

The equality $F(x,y)=0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ follows from the two equalities

$$F(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Let\`s give arbitrary values to a and b. For example, a=1 and b=2 let\`s take it. From this we create the equation

$$\frac{x^2}{1^2}+\frac{y^2}{2^2}-1=0\,,x^2+\frac{y^2}{4}-1=0.$$

Let us find a point $M_0(x_0,y_0)$ that satisfies this equation. From this we find $M_0(\frac{1}{2},\sqrt{3})$. Let\`s check this point $M_0$, against our theorem.

1) We defined the function $F(x,y)$ around the point $M_0(x_0,y_0)$ and it is equal to $M_0\left(\frac{1}{2},\sqrt{3}\right)$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

has a special derivative

2) Let the function $F(x,y)$ become zero at this point: $F(x_0,y_0)=0$; Let\`s check the point

$$x^2+\frac{y^2}{4}-1=0$$

$$M_0.\frac{1}{4}+\frac{3}{4}-1=0$$

is valid.

3) Let the derivative $F_y`(x,y)$ be different from zero from this point:

$$F_y`(x_0,y_0)\neq 0.$$

Since

$$F_y`=\frac{2y}{4}=\frac{\sqrt{3}}{2}\neq 0.$$

It fulfills the conditions of the theorem.

**Example 2.** Let\`s look at the function

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-tg\left(\frac{\pi y}{4\sqrt{3}}\right)=0$$

compared to our example 1, $a,b$ took their values. But instead of 1, we got $tg(\frac{\pi y}{4\sqrt{3}})$.In our example 1, it was possible to find $y$. That is

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

remained equal. It is very difficult to find $y$ in our function. Let\`s look at the points $a,b$ and $M_0$ in our example 1.

Let\`s put it in the terms of our theorem.

1) We defined the function $F(x,y)$ around the point $M_0(x_0,y_0)$ and equaled $M_0(\frac{1}{2},\sqrt{3})$.

2) Let the function $F(x,y)$ become zero at this point: $F(x_0,y_0)=0$;

$$x^2+\frac{y^2}{4}-tg(\frac{\pi y}{4\sqrt{3}})=0\ \ M_0$$

let\`s check the point.

$$\frac{1}{4}+\frac{3}{4}-tg(\frac{\pi\sqrt{3}}{4\sqrt{3}})=0 \qquad 1-tg(\frac{\pi}{4})=0 \quad 1-1=0$$

equality arises and is appropriate.

3) Let the derivative $F_y`(x,y)$ be different from zero at the point ahu: $F_y`(x_0,y_0)\neq 0$.

$$F_y` = \frac{2y}{4} - \frac{\frac{\pi}{4\sqrt{3}}}{cos^2\frac{\pi y}{4\sqrt{3}}} = \frac{\sqrt{3}}{2} - \frac{\frac{\pi}{4\sqrt{3}}}{cos^2\frac{\pi\sqrt{3}}{4\sqrt{3}}} = \frac{\sqrt{3}}{2} - \frac{\pi}{2\sqrt{3}} \neq 0$$

is equal and fulfills the conditions of the theorem.

## Literature:

1. M.S. Salohitdinov, O‗.N. Nasritdinov. Oddiy differensial tenglamalar. T. «Uzbekiston» , 1994 y., 103 – 155 betlar.

2. Wright J. Discrete analogues in harmonic analysis. Wroclaw (2013).

3. Архипов Г. И., Садовничий В. А., Чубариков В. Н. Лекциипо математическому анализу. —М.: Высшая школа, 2000.

4. Ильин В. А., Садовничий В. А., Сендов Б. Х. Математический анализ. В 2-х ч. Изд. 2-е, перераб. — М.: Изд-во Моск. ун-та, 1985, 1987.

5. Камынин Л. И. Курс математического анализа. В 2-х ч. — М.: Изд-во Моск. ун-та, 1993, 1995.

6. Кудрявцев Л. Д. Курс математического анализа. В 3-х т. — М.: Высшая школа, 1988, 1989.

7. Никольский С. М. Курс математического анализа. В 2-х т. — М.: Наука, 1990.

8. Виноградова И. А., Олехник С. Н., Садовничий В. А. Задачи и упражнения по математическому анализу. — М.: Изд-во Моск. ун-та,1988.

9. Демидович Б. П. Сборник задач и упражнений по математическому анализу. — М.: Наука, 1990.

10. Макаров Б. М., Голузина М. Г., Лодкин А. А., Подкорытов А. Н. Избранные задачи по вещественному анализу. — М.: Наука, 1992.

## РЕЗЮМЕ

Morse Lemmasi matematik tahlilning asosiy teoremasi bo'lib, u kollektorlardagi silliq funksiyalarning kritik nuqtalarining xatti-harakatlarini chuqur tushunish imkonini beradi. Bu lemma kritik nuqtalarning lokal tavsifini beradi, ularning barqarorligini va asosiy fazo topologiyasini ochib beradi. Ushbu maqolada biz Morse lemmasining isbotini muhokama qilamiz. Bundan tashqari, biz uning singularlik nazariyasidagi rolini o'rganamiz, bu erda u yagona nuqtalarning tasnifi va xatti-harakatlariga yoritib beradi. Qattiq matematik taqdimot va tasviriy misollar orqali biz Morze lemmasining matematik tahlilda chuqur ta'siri va foydaliligini etkazishni maqsad qilganmiz.

## РЕЗЮМЕ

Лемма Морса является фундаментальной теоремой математического анализа, предлагающей глубокое понимание поведения критических точек гладких функций на многообразиях. Эта лемма дает локальную характеристику критических точек, раскрывая их устойчивость и топологию лежащего в их основе пространства. В этой статье мы обсудим доказательство леммы Морса. Более того, мы исследуем ее роль в теории особенностей, где она проливает свет на классификацию и поведение особых точек. Посредством строгого математического изложения и наглядных примеров мы стремимся передать глубокое влияние и полезность леммы Морса в математическом анализе.

## SUMMARY

Morse's Lemma stands as a fundamental theorem in mathematical analysis, offering profound insights into the behavior of critical points of smooth functions on manifolds. This lemma provides a local characterization of critical points, revealing their stability and the topology of the underlying space. In this paper, we discuss the proof of Morse's Lemma. Moreover, we examine its role in singularity theory, where it sheds light on the classification and behavior of singular points. Through rigorous mathematical exposition and illustrative examples, we aim to convey the profound impact and utility of Morse's Lemma in mathematical analysis.