TO'RTINCHI TARTIBLI YUKLANGAN TENGLAMA UCHUN BIR ARALASH MASALANING YECHILISHI

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***Annotatsiya**: Maqolada to'rtinchi tartibli yuklangan tenglama uchun bir aralash masala qaralgan. Masalaning yechimi o'zgaruvchilarni ajratish usuli bilan topilgan. Masalaning yechimining mavjudligi, yagonaligi va turgunligi ko'rsatilgan.*

***Kalit so'zlar:** yuklangan tenglama, o'zgaruvchilarni ajratish usuli, yechimning mavjudligi, yagonaligi va turgunligi.*

Kirish. So'nggi yillarda yer osti suvlari va tuproq namligi sathini optimal boshqarish, bashorat qilish va tartibga solish masalalarining jadal tadqiq etilishi natijasida "yuklangan tenglama" deb nomlangan yangi tenglamalar sinfini o'rganish zarurati paydo bo'ldi. Birinchi marta "yuklangan tenglama" atamasi A.M.Naxushevning ilmiy ishlarida qo'llanilgan bo'lib, ularda yuklangan tenglamaning umumiy ta'rifi va turli xil yuklangan tenglamalarning batafsil tasnifi berilgan, ya'ni, yuklangan differensial, integral, integro-differensial, funksional tenglamalar, shuningdek, ularning ko'plab tatbiqlari keltirilgan. Yuklangan tenglamalar uchun aralash va chegaraviy masalalar [1, 2, 4, 5] ishlarda qaralgan. To'rtinchi tartibli tenglamalar uchun aralash va chegaraviy masalalar [3,6] ishlarda ko'rilgan.

Masalaning qo'yilishi. $\Omega=\{(x,t):0<x<1,0<t<\beta\}$ sohada

$$u_{tt}+a^2u_{xxxx}+b^2u+\mu(t)u(x,d)=F(x,t) \tag{1}$$

$$u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x), \tag{2}$$

$$u_x(0,t)=0, u_x(p,t)=0, u_{xxx}(0,t)=0, u_{xxx}(p,t)=0 \tag{3}$$

masalani qaraymiz, bunda $\varphi(x)$ va $\psi(x)$ berilgan yetarlicha silliq funksiyalar, $\varphi'(0)=\varphi'(p)=0,\ \varphi'''(0)=\varphi'''(p)=0$.

Masala yechimining mavjudligi va yagonaligi. Masala yechimini $u(x,t)=X(x)T(t)$ ko'rinishida izlaymiz. Buni tenglamaga qo'yib, o'zgaruvchilarni ajratib

$$X^{IV}(x)+\lambda X(x)=0$$

$$X'(0)=X'(p)=0,\ X'''(0)=X'''(p)=0$$

spektral masalaga ega bo'lamiz. Bu masalaning xos sonlari va xos funksiyalari mos ravishda

$$X_0(x)=\frac{1}{\sqrt{p}},\ X_k(x)=\sqrt{\frac{2}{p}}\cos\lambda_k x, \lambda_0=0,\ \lambda_k=\frac{k\pi}{p}, k=1,2,... \tag{4}$$

bo'ladi. (4) sistema ortonormallangan, to'la va $L_2(0,p)$ fazoda bazis hosil qiladi. Masalaning yechimini

$$u(x,t)=X_0(x)T_0(t)+\sum_{k=1}^{\infty}X_k(x)T_k(t) \tag{5}$$

ko'rinishda izlaymiz, bu yerda

$$T_0(t)=\int_0^p u(x,t)X_0(x)dx,\ T_k(t)=\int_0^p u(x,t)X_k(x)dx. \tag{6}$$

(6) asosida quyidagi

$$T_{0,\varepsilon}(t)=\int_{\varepsilon}^{p-\varepsilon}u(x,t)X_0(x)dx,\ T_{k,\varepsilon}(t)=\int_{\varepsilon}^{p-\varepsilon}u(x,t)X_k(x)dx \tag{7}$$

funksiyalarni kiritamiz, bunda $\varepsilon>0$ yetarlicha kichik son. (7) ni ikki marta differensiallab, (1) tenglamani hisobga olib, quyidagi tengliklarga ega bo'lamiz:

$$T''_{0,\varepsilon}(t)=\int_{\varepsilon}^{p-\varepsilon}\left[F(x,t)-a^2u_{xxxx}-b^2u-\mu(t)u(x,d)\right]X_0(x)dx,$$

$$T''_{k,\varepsilon}(t)=\int_{\varepsilon}^{p-\varepsilon}\left[F(x,t)-a^2u_{xxxx}-b^2u-\mu(t)u(x,d)\right]X_k(x)dx.$$

Bu tengliklarning o'ng tomonidagi u_{xxxx} ni o'z ichiga olgan integrallarni to'rt marta bo'laklab integrallaymiz. Chegaraviy shartlarni hisobga olib, $\varepsilon\to 0$ da limitga o'tib,

$$T_0''(t)+b^2T_0(t)=f_0(t) \tag{8}$$

$$T_k''(t)+v_k^2T_k(t)=f_k(t) \tag{9}$$

tenglamalarga ega bo'lamiz, bunda

$$f_0(t)=F_0(t)-\mu(t)T_0(d),\ f_k(t)=F_k(t)-\mu(t)T_k(d),$$

$$F_0(t)=\int_0^p F(x,t)X_0(x)dx, F_k(t)=\int_0^p F(x,t)X_k(x)dx.$$

(8) va (9) tenglamalarning mos ravishda

$$T_0(0)=\varphi_0, T_0'(0)=\psi_0 \text{ va } T_k(0)=\varphi_k, T_k'(0)=\psi_k$$

shartlarni qanoatlantiruvchi yechimlari

$$T_0(t)=\varphi_0\cos bt+b^{-1}\psi_0\sin bt+b^{-1}\int_0^t F_0(\tau)\sin b(t-\tau)d\tau-$$

$$-\frac{1}{\Delta_0(d)}\left[\varphi_0\cos bd+b^{-1}\psi_0\sin bd+b^{-1}\int_0^d F_0(\tau)\sin b(d-\tau)d\tau\right]\int_0^t \mu(\tau)\sin b(t-\tau)d\tau, \quad (10)$$

$$T_k(t)=\varphi_k\cos v_k t+v_k^{-1}\psi_k\sin v_k t+v_k^{-1}\int_0^t F_k(\tau)\sin v_k(t-\tau)d\tau-$$

$$-\frac{1}{\Delta_k(d)}\left[\varphi_k\cos v_k d+v_k^{-1}\psi_k\sin v_k d+v_k^{-1}\int_0^d F_k(\tau)\sin v_k(d-\tau)d\tau\right]\int_0^t \mu(\tau)\sin v_k(t-\tau)d\tau \quad (11)$$

bo’ladi, bunda

$$v_k=\sqrt{a^2\lambda_k^4+b^2},\ \varphi_0=\int_0^p\varphi(x)X_0(x)dx,\varphi_k=\int_0^p\varphi(x)X_k(x)dx,$$

$$\psi_0=\int_0^p\psi(x)X_0(x)dx,\psi_k=\int_0^p\psi(x)X_k(x)dx,$$

$$\Delta_0(d)=b+\int_0^d\mu(\tau)\sin b(t-\tau)d\tau,\ \Delta_k(d)=v_k+\int_0^d\mu(\tau)\sin v_k(t-\tau)d\tau\,.$$

Endi masala yechimining yagonaligini ko’rsatamiz. Faraz qilaylik, $\varphi(x)\equiv 0, \psi(x)\equiv 0$ bo’lsin, unda $\varphi_k\equiv 0, \psi_k\equiv 0$ bo’ladi. (6) tenglikdan va (10), (11) yechimlardan

$$\int_0^p u(x,t)X_0(x)dx=0,\ \int_0^p u(x,t)X_k(x)dx=0$$

tengliklarga ega bo’lamiz. (4) funksiyalar sistemasining $L_2(0,p)$ fazoda to’laligidan va $u(x,t)$ funksiyaning uzluksizligidan, $\bar{\Omega}$ sohada $u(x,t)\equiv 0$ ekanligi kelib chiqadi.

Endi masala yechimining mavjudligini ko’rsatamiz. [6] da ba’zi bir shartlarda $\Delta_0(d)\geq c_0 b^2,\ \Delta_k(d)\geq c_1 k^2$ ekanligi ko’rsatilgan.

1-lemma. Katta *k* natural sonlar uchun quyidagi baholashlar o’rinli bo’ladi:

$$|T_k(t)|\leq c_2\left(|\varphi_k|+\frac{1}{k^2}|\psi_k|+\frac{1}{k^2}\|F_k(t)\|\right),$$

$$\left|T_k'(t)\right| \le c_3\left(k^2|\varphi_k| + |\psi_k| + \|F_k(t)\|\right),$$

$$\left|T_k''(t)\right| \le c_4\left(k^4|\varphi_k| + k^2|\psi_k| + k^2\|F_k(t)\| + |F_k(t)|\right).$$

2-lemma. Faraz qilaylik, $\varphi(x)\in C^5[0,p], \psi(x)\in C^3[0,p]$, $\varphi^{(2i-1)}(0)=\varphi^{(2i-1)}(p)=0,\ \ i=1,2;\ \ \psi'(0)=\psi'(p)=0,$ $F(x,t)\in C^{3,0}_{x,t}(\bar{\Omega})$, $F_x(0,t)=F_x(p,t)=0$ shartlar bajarilsin. U holda

$$\varphi_k = -\frac{1}{\lambda_k^5}\bar{\varphi}_k^{(5)}, \psi_k = \frac{1}{\lambda_k^3}\bar{\psi}_k^{(3)}, F_k(t) = \frac{1}{\lambda_k^3}\bar{F}_k^{(3)}(t) \tag{12}$$

tengliklar to’g’ri bo’ladi, bunda

$$\bar{\varphi}_k^{(5)} = \sqrt{\frac{2}{p}}\int_0^p \varphi^V(x)\sin\lambda_k x dx,\ \bar{\psi}_k^{(3)} = \sqrt{\frac{2}{p}}\int_0^p \psi'''(x)\sin\lambda_k x dx,$$

$$\bar{F}_k^{(3)}(t) = \sqrt{\frac{2}{p}}\int_0^p F_{xxx}(x,t)\sin\lambda_k x dx.$$

1-teorema. Faraz qilaylik, $\varphi(x), \psi(x)$ va $F(x,t)$ funksiyalar 2-lemmaning shartlarini qanoatlantirsin. U holda (1)-(3) masalaning yechimi mavjud bo’ladi va bu yechim (5) qator ko’rinishida bo’ladi.

Isboti. (5) qatorni t bo’yicha ikki marta, x bo’yicha to’rt marta hadma-had differensiallaymiz:

$$u_{tt}(x,t) = X_0(x)T_0''(t) + \sum_{k=1}^{\infty} X_k(x)T_k''(t),\ u_{xxxx}(x,t) = \sum_{k=1}^{\infty}\lambda_k^4 X_k(x)T_k(t). \tag{13}$$

Bu qatorlar uchun majoranta qatori

$$c_4\sum_{k=1}^{\infty}\left(k^4|\varphi_k| + k^2|\psi_k| + k^2\|F_k(t)\| + |F_k(t)|\right) \tag{14}$$

bo’ladi. (12) asosida (14) dan quyidagiga ega bo’lamiz:

$$c_4\sum_{k=1}^{\infty}\frac{1}{k}\left(\left|\bar{\varphi}_k^{(5)}\right| + \left|\bar{\psi}_k^{(3)}\right| + \left\|\bar{F}_k^{(3)}(t)\right\| + \frac{1}{k^2}\left|\bar{F}_k^{(3)}(t)\right|\right).$$

Bu qator yaqinlashuvchi bo’ladi. U holda Veyershtrass teoremasi bo’yicha (14) qatorning yaqinlashuvchiligidan (5) va (13) qatorlarning tekis yaqinlashuvchanligi kelib chiqadi. Teorema isbotlandi.

Masalaning yechimining turg’unligi. Quyidagi normani kiritamiz:

$$\|u(x,t)\|_{L_2[0,p]} = \left(\int_0^p |u(x,t)|^2 dx\right)^{\frac{1}{2}}.$$

2-teorema. Aytaylik, 1-teoremaning shartlari bajarilsin. U holda quyidagi baholash o’rinli bo’ladi:

$$\left\|u(x,t)\right\|_{L_2(0,p)} \le c\left(\left\|\varphi\right\|_{L_2(0,p)} + \left\|\psi\right\|_{L_2(0,p)} + \left\|F(x,t)\right\|_{L_2(\Omega)}\right).$$

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